Physical Principles of Rocketry
A Practical Guide

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1 Introduction

The problem of basic rocket dynamics and stability has been discussed extensively and several elementary treatments by Stine[1] and others are readily available. What we'll do here is develop a first principles mechanical understanding of rocket propulsion and then an easily applied quantitative model for a model rocket's performance based on that understanding.

2 Propulsion

Most discussions of the rocket engine are based on an accurate, but almost legalistic, incantation of Newton's laws of motion as applied to a balloon spewing gas or some such example. In the students mind, a common result is (at least) the opinion that rockets do indeed work for good and sensible reasons. While a good start, such an opinion can be developed readily into a clear physical model — a means to predict the path, speed, and duration of any model's flight from a few simple data.

When posed correctly the trajectory problem for a single stage rocket of known mass, propellant, and cross section can be reduced to a common rule that is relatively independent of the construction details and environmental perturbations. The scope of this discussion is therefore the derivation and application of this *"Simplest Relevant Performance Model"*.

2.1 Elements of Mechanics

How *much* motion does an object have? What are the simplest properties required to describe and to predict that motion? These questions are intimately linked, because the measure of motion, a length per unit time $[\ell/t]$ quickly shows that the production of motion is inhibited by a property that is independent of motion, the inertia or mass $[m]$ of any object. Moreover the inertia is the same no matter which direction the motion takes, and the inertia is

additive as well. Two crates, one atop the other, on a (uniformly) slick floor are twice as hard to move, in *any* direction, as just one such crate. If we are to assign a unique set of measures to an object moving in one, two, or three directions, then such measures must scale in proportion to the mass but be otherwise independent as to the direction. Why independent? Well we know that we can direct impulses to change motion in such a way as to leave motions in perpendicular directions unaffected — the parabolic trajectory of the tossed slug being the most common example. With gravity controling the motion vertically but nothing forcing it along the ground, the quantity of motion we infer for the vertical coordinate is just not coupled to that for the horizontal coordinates.

A similar action is portrayed in the figure below, where a ball of fixed mass rolls along a "billiard table with a small hill". The motion is divided along any of three independent directions (as labeled) and all forces are directly

The Quantity of Motion: $P = p_1 + p_2 + p_3$
must be a vector and linear in the mass
and velocity, so $p_k = m\, v_k$.
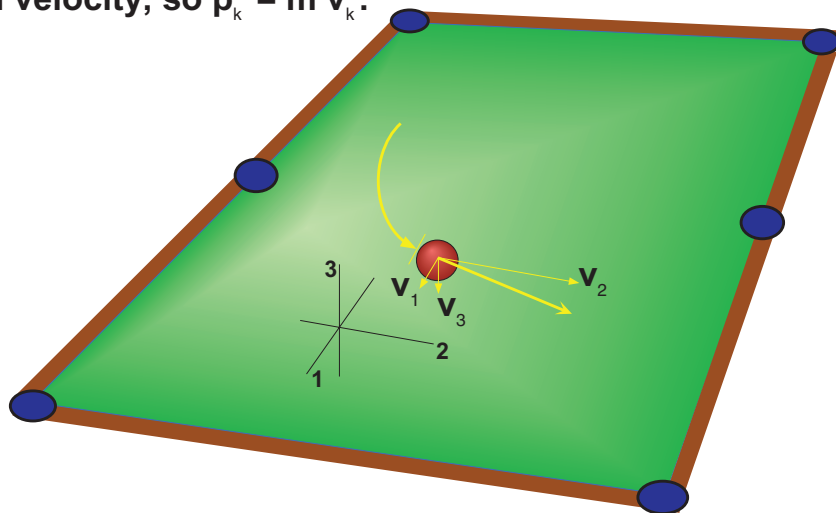
influenced by gravity. From the path shown, horizontal impulses must occur. These are determined by the constraint forces supplied by the surface prevent the mass from falling through it. The changes in the momentum vector **P** as the motion progresses are determined by the independent tilt angles along the "1" and "2" directions. If the table is very stiff in its response to the weight of the mass, then no change of kinetic energy will be seen in the horizontal motion. In contrast the kinetic energy in the verticle motion diminishes and then rebounds as the ball climbs the hill and rolls over it. Clearly the **only** vector we can use as the measure of motion is the linear momentum vector

given — other combinations such as $mv_k v^2$, or $mv_k v_j^2$ violate the requirement that the action must be allowed to develop independently along the three coordinates. Any combination of mass and velocity components other than a product will violate the observation that inertia must be proportional and independent of the direction of motion.

The vector $\mathbf{P} \equiv m\left(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\right)$ is therefore the *only* admissible quantity as a primary variable in mechanics. Finding the motion $[\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)]$ to be expected from a known set of changes in this momentum is then the fundamental task of mechanics and the simple rule for doing this is the statement that:

$$\mathbf{F}_{ext} \equiv \frac{d\mathbf{P}}{dt} \ , \tag{1}$$

an embodiment of Newton's First and Second Laws. The operational definition of an "external" force is any influence capable of generating a change in the momentum vector. The forces dependent on the angle of the hill in the billiard table example are "external" too, but further specialized. They are perfect examples of constraint forces — the tensile strength of the table must arise *as needed* to prevent the penetration of the table. The motion of a rocket depends on something a lot less subtle.

So back to our billiard table again, to replace the ball with some device that expends mass "downward" and moves upward along the $\mathbf{e}_3$ axis. For any length of time we can account for an amount of mass $m_e(t)$ that has been expended at an average velocity $\mathcal{U}$, the exact time history of the mass loss is arbitrary, viz. we do not need to decide on its time dependence now. The average speed $\mathcal{U}$ can always be found so that the expended momentum $p_e(t) = m(t)U(t)$ is exactly equal to our mass history function multiplied, by the average speed, viz. $\exists \mathcal{U} : p_e(t) = \mathcal{U} m_e(t)$. Application of Eqn (1) to this situation provides,

$$\mathbf{F}_{ext,3} = \frac{d((m_o - m_e)\mathbf{v}_3 + m_e(\mathbf{v}_3 - \mathbf{e}_3\mathcal{U}))}{dt} \ , \tag{2}$$

where in the second term the speed of exhaust (as we observe it) is just the average speed $\mathcal{U}$ (seen from the device) diminished by v_3 (the speed at which the device rises). Using the mass history to play the role of the momentum history will be a good approximation so long as the true exhaust speed history is "close" to the average or departs from it only for short fragments of time. If a large fraction of mass is ultimately expended, the errors in the inertia produced by $m_e(t)$ will be more noticeable but as we shall presently show the sensitivity of the trajectory to this error is weak.

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Writing $m_r = m_o - m_e$, using the external force $-gm_r$, and expanding the derivative operation provides the result

$$\frac{d\mathbf{v}_3}{dt} = \mathbf{e}_3 \left(\mathcal{U} \, (\frac{-1}{m_r} \frac{dm_r}{dt}) - g \right) \, , \tag{3}$$

which can be immediately integrated to provide the well known "rocket equation"

$$v(t) = v_o - gt + \mathcal{U} \, \ln \left(\frac{1}{1 - \frac{m_e(t)}{m_o}} \right) \, . \tag{4}$$

The velocity is seen to be linearly sensitive to getting the right average exhaust speed, but only logarithmically sensitive to the fraction of mass expended. In most solid fuel boosters and certainly for those used in model rockets, the exhaust velocity is kept roughly constant while the shape of the burn surface determines the mass ejection history. Note also that when the mass ejection fraction (m_e/m_o) rises to 0.632120559 the rocket velocity will equal the average exhaust speed $\mathcal{U}$. A mass ejection fraction (MEF) of 0.864664717 is required to double the exhaust speed. Practical rocket systems almost never achieve this kind of performance, as a consequence the primary arena for engineering improvements lies in the achievement of a high exhaust speed. The fraction of final mass to initial mass is of course just $1 - \frac{m_e}{m_o}$.

What are some typical values for $\mathcal{U}$ in real model rockets? Well most model rocket engines have MEF values ranging from 0.189 for A series engines up to 0.58 for D series engines. In other words a D engine (mass about 43 g) launched alone with no air resistance and no retarding gravity would achieve about 0.87 its exhaust velocity. We can readily estimate that exhaust velocity to be:

$$\mathcal{U}_D = \frac{20Ns}{0.043kg \ln(1/1 - MEF)} \approx 536[m/s] \, ,$$

so the burnout velocity of a D motor in *free space* would be 465 [m/s] for a Mach number of 1.4 — not bad for a toy! Examining the typical motors used in model rocketry, the scale velocity for each motor class is given in Table 1. Also in this table are the typical burn times (τ_b) and expended propellant masses m_e for the various engines. Now for any class engine we can estimate the average acceleration for real model to be

$$\bar{a}_i \approx \frac{\mathcal{U}_i}{\tau_b} \frac{m_{eng}}{m_{rocket}} [m/s^2] \, .$$

Table 1: Engines — Scale Velocities, Burn Times, and Expended Mass

$\mathcal{U}_A = 723$ [m/s] | $\tau_b = 0.24$ [s] | $m_e = 3.12$ [gm]
$\mathcal{U}_B = 668$ [m/s] | $\tau_b = 0.82$ [s] | $m_e = 6.24$ [gm]
$\mathcal{U}_C = 587$ [m/s] | $\tau_b = 1.70$ [s] | $m_e = 12.50$ [gm]
$\mathcal{U}_D = 536$ [m/s] | $\tau_b = 1.70$ [s] | $m_e = 24.95$ [gm]

A further simplification arises if we study the rated impulse in Newton seconds compared with the mass of propellant m_e, as listed in the manufacturers notes that come with each set of engines purchased. Common to all black powder engines is *one number*, $\mathcal{I}_{sp} = I_{rated}/m_e g = 81$ [s]. The impulse per unit weight of propellant, or *specific impulse* is a characteristic figure of merit for the fuel itself. Needless to say rocket science is reduced in practice to obtaining the best possible specific impulse fuels. The best liquid fuels like the hydrogen-oxygen mix burned by the shuttle have specific impulse of 450 to 550 [s], ion thrusters raise that to about 3000 [s], and nuclear fusion engines may obtain 10000 [s].

2.2   Equation of Motion For A Real Rocket

In the examples above the rocket motion was simplified in two important ways. First the effect of gravity was not included past Eqn (4) in the determination of the scale velocity boost expected from the engines treated. Second the aerodynamic drag was not included in any of the estimates. The great advantages of those simplifications have now been exploited and "boiled down" to the contents of Table 1. Hence to treat the real world model rocket it is time to face into the wind and seek a treatment of the drag forces which is at once comprehensive and tractable.

Drag forces are found to scale with (1) the mass density of the ambient medium (1.1764 kg/m^3 for a standard atmosphere of air at 300 ^oK), (2) the area presented normal to the flow stream ($\pi d_o^2/4$ for a cylinder d_o in diameter), and (3) the square of the velocity through the medium. If an object is pulled through such a medium by gravity alone, then it will reach a terminal velocity when the drag force exactly balances the pull of gravity. For given rocket airframe of diameter d_o and mass m_r this *terminal velocity* v_t can be calculated by writing the balance condition for gravity to equal drag, viz.

$$m_r g = c_D \left(\frac{\rho \pi d_o^2}{8} \right) v_t^2 \ , \tag{5}$$

where we have introduced the body averaged dimensionless *drag coefficient* c_D as a means of accounting for the detailed shape of the object. The value of c_D ranges from about one to a minimum of a few percent. It can exceed one, but not by much, and a typical value for a model rocket airframe is about 0.75 or a bit less. Computing a drag coefficient from first principles is a tedious business, so they are usually measured. Hiding in c_D is a weak depencence on the angle of attack between the body axis and the flow direction and a further dependence on the Reynolds number $\mathcal{R}_n$ of the flow. The Reynolds number is again a dimensionless parameter, it derives from the viscosity of the fluid and the fluid inertia. From a practical viewpoint we can just remember to keep c_D in a sensible range and treat it as the adjustable engineering parameter it truly is — for the most part a winning model rocket gets the lowest c_D it can for a given configuration of tubes, fins, mass, and surface treatments.

If we want to calculate a model rocket's trajectory, we simply treat v_t as a specified parameter. The terminal velocity will increase with the rocket mass, decrease with increasing c_D, and serve to set the scaling of the drag force as the rocket velocity increases. Since drag force equals mg at v_t and varies as the square of velocity, the retarding drag force expression is just $g(v/v_t)^2$. The equation of motion for the vertical coordinate then balances the positive thrust of the engine against the retarding forces of drag and gravity,

$$\frac{dv}{dt} = \left(\frac{\mathcal{U}}{\tau_b}\right)\left(\frac{-\tau_b}{m_r}\frac{dm_r}{dt}\right)_S - g\left(\frac{v}{v_t}\right)^2 - g, \qquad (6)$$

and the whole action is determined by: (1) the scale velocity of the engine $\mathcal{U}$, (2) the burn time τ_b, (3) the standard acceleration of gravity g, (4) the terminal velocity v_t, (5) the mass of the rocket $m_r = m_o - m_e$, and (6) the dimensionless thrust shape factor for the engine $S(t)$. Here it is reasonable to take $c_D \approx 24/\mathcal{R}_n \leq 0.75$ and to evaluate the terminal velocity $v_t^2 = (212.28 \, m_{o,gm}/c_D d_{o,cm}^2) \, [m/s]^2$ on the basis of the initial mass. The small variations in the drag coefficient and the rocket's mass distribution during the flight will change this drag strength parameter and hence the trajectory as well, but probably at insignificant levels.

As to the thrust shape factor, with no loss of generality, one may set $S(t) = \tau_b d/dt \ln(1/m_r(t))$ as an accurate shape function for a fixed or average mass loss rate, $m_e(t) = (t/\tau_b)m_e$. The detailed calculation for the shape function provides

$$S(t) \equiv \left(\frac{-\tau_b}{m_r}\frac{dm_e(t)}{dt}\right) = \frac{m_e/m_o}{1 - (t/\tau_b)m_e/m_o}, \qquad (7)$$

and, while we could use others, such as one involving a quadratic time dependence in the case of the linear increase in mass loss rate from a core burner engine, the general predictions of the model will not be very sensitive to these details. After the interval τ_b the shape factor goes to zero and the rocket just drags and coasts to peak altitude.

In the figure below — "A Tale of Three Rockets" — are numerically computed velocity solutions (green curves, scale is m/s) and altitude solutions (red curves, scale is m) for three similar model rockets operating on D impulse engines. In the first case (A), the rocket mass is 100 gm and the airframe is about an inch in diameter with an assumed drag coefficient of 0.75. These conditions conspire to require a terminal velocity for the airframe of about 66.2 m/s. In the second case (B) the mass is increased to 200 gm, increasing the terminal velocity, but lowering the MEF from 0.25 to 0.125. In the third case we recover the good MEF by strapping in two D engines, at a cost of twice the diameter for the airframe, decreasing the terminal velocity. The extra drag is more than compensated by the extra lift so that the anemic performance of the second case is erased with an altitude exceeding the first shot.
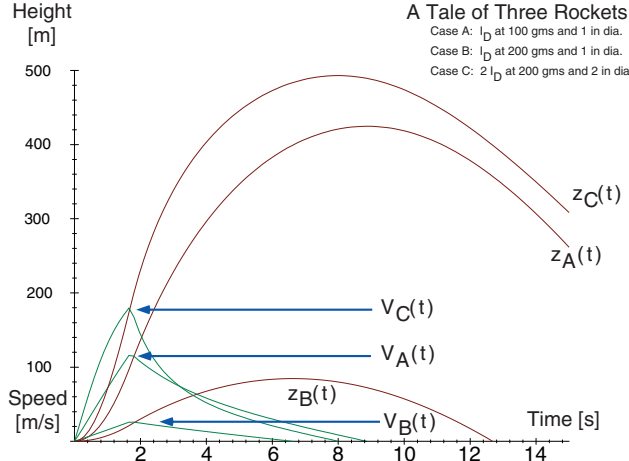
Clearly the maintainance of a proper impulse relative to mass and the use of a larger MEF in the power plant are key ingredients to achieving high altitudes. Note that for case C especially the velocity curve is beginning to "roll over" near burnout, indicating the approach to a terminal velocity on the way up! If the same engines were equipped to burn longer at the same rate, then an asymptotic terminal velocity would be set by the balance of net upward thrust and drag. The final velocity would be constant in time and larger than the parameter v_t used to scale the drag for a falling model.

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Table 2: Variation of performance with drag coefficient and MEF

C Impulse: 100 gm rocket				D Impulse: 200 gm rocket			
C_d	t_{max}	Z_{max}	V_{max}	C_d	t_{max}	Z_{max}	V_{max}
1.00	5.12	114.27	49.35	1.00	5.66	129.22	49.23
0.75	5.46	128.16	51.83	0.75	5.93	139.96	50.50
0.50	5.93	148.10	54.66	0.50	6.26	153.67	51.85
0.25	6.65	180.12	57.90	0.25	6.69	172.06	53.29
MEF	t_{max}	Z_{max}	V_{max}	MEF	t_{max}	Z_{max}	V_{max}
0.1250	5.12	114.27	49.35	0.1250	5.66	129.22	49.23
0.1875	5.77	180.17	74.90	0.1875	6.85	231.06	79.45
0.2500	6.11	234.17	97.96	0.2500	7.51	319.91	108.57
0.3125	6.33	281.09	119.73	0.3125	7.93	398.01	137.19

2.3   The Simplest Relevant Performance Model

From the calculation given above in the three rocket case it should be clear by now that all essential features of the motion are governed by two factors. A greater mass ejection fraction for any total impulse rating always leads to faster speeds and higher altitudes. Similarly a lower drag coefficient moves the performance in the same direction. Once a rocket is mated with the proper total impulse for its mass, the two *dimensionless numbers* MEF and c_D pretty much determine what optimizations you can squeeze out of the design.

So again we return to the numerical solution of the equation of motion (6) to examine the consequences of variations in MEF and c_D. Running several cases for two "typical" rockets provide the data in Table 2.

On the left of the table is data for a standard single C engine on a 100 gm rocket, 2 in. diameter, to yield v_t=50.0/$\sqrt{2}$. On the right of table, similar data for a standard single D engine on a 200 gm rocket, also 2 in. diameter which increases v_t=50.0. On both test cases the mass ejection fraction (or MEF) varies from 0.125 ... 0.3125; while the drag coefficient (c_D) varies from 1.0 ... 0.25.

As you study this table notice that MEF has a decidedly stronger effect on performance than drag adjustments. This is because almost all model rocket motors are on the sensitive side of the performance curve, viz. they do not even come close to achieving a burnout velocity near the exhaust velocity.

If you stick to the typical black powder engines, then any design games you might play will always place your rocket somewhere on this table, where

you can find a reasonable upper bound for the altitude and an estimate of the time to peak altitude which will allow you to choose a proper engine delay for the recovery system of your choice.

3   Stability

Up to now the "rocket" in our mathematical model has been treated as a point mass. When we build a model of the drag force that looks like $g(v/v_t)^2$, it is a statement that the sum of all the detailed pushes from the air flow at various points of the rocket's airframe can act as if they are concentrated at a single point in space. What special point is this? Are we sure that every rocket has one?

Well the point is called "center of mass" and *every* chunk of matter, from the Andromeda galaxy to the Queen Mary to the family cat, has one. The first figure below illustrates how you determine it, as the center of gravity in an operational sense. As shown a given mass at twice the distance is equal in effect to twice that same mass at a given distance, i.e. the assembly will balance when you find the center of gravity.

The Center of Gravity
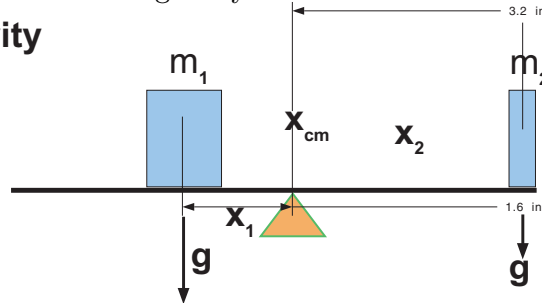
The Center of Pressure
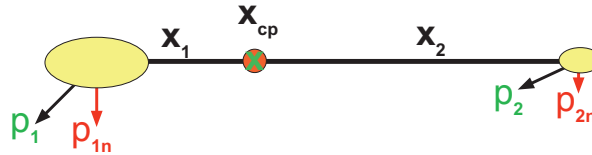
In the second figure we define a similar point called "center of pressure". The idea here is simply to replace gravity force with air force perpendicular to the object's surface. If the pressure times distance on the left is equal to the pressure times distance on the right, then the assembly as shown will not

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"twist in the wind" if held loosely at this center of pressure. Stability for a rocket, as we shall presently demonstrate, depends upon the relative position of these two centers.

To understand how this works, think of the drag forces as the sum of two parts relative to the rocket's momentum vector. One part must always point along the momentum vector. If the angle of attack[2] is zero, then there is only this part of the drag force. On the other hand, if the angle of attack happens to change from zero, then some part of the drag force is going to be perpendicular to the momentum vector. The drag force along the momentum vector is what we model with our drag term in the equation of motion (6). The perpendicular part will tend to twist the rocket around its center of mass.

First consider the effect of having the center of pressure coincide with the center of mass. If the rocket tilts off the vertical, then the drag forces perpendicular to its upward momentum vector will balance exactly. The tilt will not be restored and hence the rocket will be pushed into the new direction that its nose has drifted into. This situation is termed *neutral stability*.

Next consider the result of putting the center of pressure ahead of the center of mass. Now a deviation in the angle of attack will be amplified by the perpendicular drag force, the rocket will in effect glance off the air mass in front of it and take on a new direction at greater angle of attack. What is observed can be chaotic twists and turns as the rocket continually tumbles over its center of pressure and assumes a new course. This situation is termed *unstable*.

Finally consider the effect of putting the center of pressure behind the center of mass. Any change in the angle of attack away from zero is met with a restoring force that drives the tail back toward the original momemtum vector or back to a zero angle of attack. The faster the rocket is going, the stronger the restoring force — recall that, being a drag force, it is scaling like the velocity squared. This situation is termed *stable*. Nice little cartoons of these three classes of motion are given by Stine[3].

In order to produce a safe model, it is imperative that the center of pressure be well behind the center of mass by a body tube diameter or more. A second basic ingredient for a safe flight is the launch rod or rail which serves to constrain the rocket's path until it gets enough velocity that drag force stabilization can "kick in". If is for this reason that many of the smaller A impulse engines are cored out to give an enhanced mass ejection rate at the start of the burn. The design gives a quick jump to a few tens of meters per second for an air frame with a small terminal velocity and forces the drag to be important as soon as the rocket leaves the rail.

4 Advanced Topics and Problems

For those inclined to mathematical modeling, a good exercise is to (1) extend the equation of motion (6) we have used here to treat a rocket of N stages and then (2) build a simple computer program to integrate the equation of motion for cases of two stage rockets to get an idea of what mass staging choices are good, bad, or indifferent.

Another useful exercise is to compile the data and estimate a table similar to our first one for the "perchlorate" engines. What do you infer for the I_{sp} of these engines?

Finally, it has been stated without proof[4] that the center of pressure for a rocket airframe with any combination of fins, body diameters, and other parts symmetrically arranged about the long axis can be inferred by tracing or projecting the three dimensional object onto a plane and then finding the center of mass for the projected shape, assuming uniform mass density. Prove (or deny) this theorem, with or without the techniques of calculus.

References

[1] G. Harry Stine, *Handbook of Model Rocketry*, 6th ed., Wiley, NY 1994

[2] ibid, p.125

[3] ibid, p.143

[4] John McCoy, private communication in a talk to Howard Squadron in October, 1999.